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Properties of a Complete Table of Symmetric Functions.

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§1. In Vol. V, *American Journal of Mathematics*, Mr. Durfee has set forth the only complete and perfectly arranged table of symmetric functions in existence.

I propose to establish some remarkable features of such a tabulation which, so far as my knowledge extends, have not as yet been noticed.

If we represent the symmetric functions by partitions in $(\)$ and the literal products by partitions in $(\)'$, Mr. Durfee's table exhibits each partition $(\)$ in terms of the partitions $(\)'$, and inversely each partition $(\)'$ in terms of the partitions $(\)$; say these constitute the first and second portions of the table; the secondary diagonal of the square is mainly composed of units, in the exceptional cases a zero replacing a unit. The first and second portions of the table lie, in the main, above and below the secondary diagonal; the fact that this is not invariably the case being entirely due to the peculiar properties of the self-conjugate partitions; in both the 12th and 13th tables there are three such, with the consequence that the corresponding portion of the secondary diagonal becomes twisted about its middle point into coincidence with the principal diagonal of the square.

The terminal units, whether lying in a principal or secondary diagonal of the square, are common to both portions of the table in such wise that any unit, together with the numbers in the same row or column lying left of it or above it, belong to the first portion, whilst the same unit, together with the numbers lying in the same row or column to the right of it or below it, belong to the second portion.

It is important to observe that terminal units must lie either in the secondary or principal diagonal, and those in the principal diagonal correspond invariably to self-conjugate partitions.

The number of partitions of the weight being uneven, the number of self-

conjugate partitions is also uneven, and in this case one of the terminal units, corresponding to a self-conjugate partition, is at the point of intersection of the two diagonals, the remaining units being symmetrically distributed on the principal diagonal in adjacent squares; on the other hand, if the number of partitions of the weight be even, so also is the number of self-conjugate partitions; and then there is no place for a number at the intersection of the diagonals, the self-conjugate units being now none of them in the secondary diagonal, but symmetrically placed about it in adjacent squares.

§2. In the Quarterly Journal of Pure and Applied Mathematics, No. 85, 1886, I gave the complete statement of the Cayley-Betti law of symmetry; viz. if any two results of the same weight be

$$\sum A_{\lambda'\mu^m\dots}(\lambda^l\mu^m\dots) = \sum A'_{\lambda''\mu'^m\dots}(\lambda''\mu'^m\dots),$$

$$\sum B_{\lambda''\mu'^m\dots}(\lambda''\mu'^m\dots) = \sum B'_{\lambda^l\mu^m\dots}(\lambda^l\mu^m\dots),$$

then

$$\sum A_{\lambda^l\mu^m\dots}B'_{\lambda''\mu'^m\dots} = \sum A'_{\lambda''\mu'^m\dots}B_{\lambda^l\mu^m\dots}$$

I proceed to the application of this theorem.

§3. Regarding the whole square as a matrix of order equal to the number of partitions of the weight, I consider a minor square matrix of any order whose secondary diagonal is coincident with that of the whole matrix; and for clearness I first suppose such a minor matrix to be situated so that its secondary diagonal contains only units, or, what is the same thing, so that it does not intersect or include the square matrix whose principal diagonal consists entirely of units.

Represent any such minor of order s by

α_{11}	α_{12}	α_{13}	$\dots$	$\alpha_{1,s-5}$	$\alpha_{1,s-4}$	$\alpha_{1,s-3}$	$\alpha_{1,s-2}$	$\alpha_{1,s-1}$	1	,
α_{21}	α_{22}	α_{23}	$\dots$	$\alpha_{2,s-5}$	$\alpha_{2,s-4}$	$\alpha_{2,s-3}$	$\alpha_{2,s-2}$	1		α_{2s} ,
α_{31}	α_{32}	α_{33}	$\dots$	$\alpha_{3,s-5}$	$\alpha_{3,s-4}$	$\alpha_{3,s-3}$	1	$\alpha_{3,s-1}$		α_{3s} ,
α_{41}	α_{42}	α_{43}	$\dots$	$\alpha_{4,s-5}$	$\alpha_{4,s-4}$	1	$\alpha_{4,s-2}$	$\alpha_{4,s-1}$		α_{4s} ,
α_{51}	α_{52}	α_{53}	$\dots$	$\alpha_{5,s-5}$	1	$\alpha_{5,s-3}$	$\alpha_{5,s-2}$	$\alpha_{5,s-1}$		α_{5s} ,
α_{61}	α_{62}	α_{63}	$\dots$	1	$\alpha_{6,s-4}$	$\alpha_{6,s-3}$	$\alpha_{6,s-2}$	$\alpha_{6,s-1}$		α_{6s} ,
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\alpha_{s-1,1}$	1	$\alpha_{s-1,3}$	$\dots$	$\alpha_{s-1,s-5}$	$\alpha_{s-1,s-4}$	$\alpha_{s-1,s-3}$	$\alpha_{s-1,s-2}$	$\alpha_{s-1,s-1}$		$\alpha_{s-1,s}$,
1	α_{s2}	α_{s3}	$\dots$	$\alpha_{s,s-5}$	$\alpha_{s,s-4}$	$\alpha_{s,s-3}$	$\alpha_{s,s-2}$	$\alpha_{s,s-1}$		α_{ss} ,

wherein the α 's and the a 's belong respectively to the first and second portions of the table.

Applications of the law of symmetry enable us to write down the $s-1$ relations:

$$\begin{aligned} \alpha_{1, s-1} + a_{2s} &= 0, \\ \alpha_{1, s-2} + a_{3, s-1}\alpha_{1, s-1} + a_{3s} &= 0, \\ \alpha_{1, s-3} + a_{4, s-2}\alpha_{1, s-2} + a_{4, s-1}\alpha_{1, s-1} + a_{4s} &= 0, \\ \alpha_{1, s-4} + a_{5, s-3}\alpha_{1, s-3} + a_{5, s-2}\alpha_{1, s-2} + a_{5, s-1}\alpha_{1, s-1} + a_{5s} &= 0, \\ \alpha_{1, s-5} + a_{6, s-4}\alpha_{1, s-4} + a_{6, s-3}\alpha_{1, s-3} + a_{6, s-2}\alpha_{1, s-2} + a_{6, s-1}\alpha_{1, s-1} + a_{6s} &= 0, \\ &\dots\dots\dots \\ \alpha_{12} + a_{s-1, 3}\alpha_{13} + \dots + a_{s-1, s-5}\alpha_{1, s-5} + a_{s-1, s-4}\alpha_{1, s-4} + a_{s-1, s-3}\alpha_{1, s-3} \\ &\quad + a_{s-1, s-2}\alpha_{1, s-2} + a_{s-1, s-1}\alpha_{1, s-1} + a_{s-1, s} = 0, \\ \alpha_{11} + a_{s2}\alpha_{12} + a_{s3}\alpha_{13} + \dots + a_{s, s-5}\alpha_{1, s-5} + a_{s, s-4}\alpha_{1, s-4} + a_{s, s-3}\alpha_{1, s-3} \\ &\quad + a_{s, s-2}\alpha_{1, s-2} + a_{s, s-1}\alpha_{1, s-1} + a_{ss} = 0, \end{aligned}$$

which are very convenient for purposes of verification. From them is deduced

$$\alpha_{11} = \begin{vmatrix} -a_{2s} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 \\ -a_{3s} & 0 & 0 & \dots & 0 & 0 & 0 & 1 & a_{3, s-1} \\ -a_{4s} & 0 & 0 & \dots & 0 & 0 & 1 & a_{4, s-2} & a_{4, s-1} \\ -a_{5s} & 0 & 0 & \dots & 0 & 1 & a_{5, s-3} & a_{5, s-2} & a_{5, s-1} \\ -a_{6s} & 0 & 0 & \dots & 1 & a_{6, s-4} & a_{6, s-3} & a_{6, s-2} & a_{6, s-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -a_{s-1, s} & 1 & a_{s-1, 3} & \dots & a_{s-1, s-5} & a_{s-1, s-4} & a_{s-1, s-3} & a_{s-1, s-2} & a_{s-1, s-1} \\ -a_{ss} & a_{s2} & a_{s3} & \dots & a_{s, s-5} & a_{s, s-4} & a_{s, s-3} & a_{s, s-2} & a_{s, s-1} \end{vmatrix} \div \Delta,$$

where Δ is a determinant of order $s-1$ whose secondary diagonal consists entirely of units and having nothing but zeros above this diagonal; so that

$$\Delta = (-)^{s+s-1+s-2+\dots+2} = (-)^{\frac{1}{2}s(s+1)-1};$$

and in the numerator, if we perform on the columns a cyclical substitution, so as to make the first column the last and then change the sign of the last column, we shall obtain a determinant multiplied by $(-)^{s-1}$; whence the sign of the fraction becomes

$$(-)^{s-1+\frac{1}{2}(s)(s+1)-1} = (-)^{\frac{1}{2}s(s-1)},$$

and accordingly

$$\alpha_{11} = (-)^{\frac{1}{2}s(s-1)} \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 & a_{2s} \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{3, s-1} & a_{3s} \\ 0 & 0 & 0 & \dots & 1 & a_{4, s-2} & a_{4, s-1} & a_{4s} \\ 0 & 0 & 0 & \dots & a_{5, s-3} & a_{5, s-2} & a_{5, s-1} & a_{5s} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_{s-1, 3} & a_{s-1, 4} & \dots & a_{s-1, s-3} & a_{s-1, s-2} & a_{s-1, s-1} & a_{s-1, s} \\ a_{s2} & a_{s3} & a_{s4} & \dots & a_{s, s-3} & a_{s, s-2} & a_{s, s-1} & a_{ss} \end{vmatrix};$$

but the matrix of this determinant is, as regards the original matrix, none other than the matrix complementary to α_{11} , in which all the α 's are put equal to zero; further, we shall find α_{ij} equal to the product of $(-)^{\frac{1}{2}s(s-1)+1+j}$ and the determinant of its complementary matrix, with the above convention as regards zeros.

Again, consider any α element in the ρ^{th} row and κ^{th} column $\alpha_{\rho\kappa}$, and in conjunction therewith the matrix obtained from the original matrix by deletion of the first $\rho - 1$ rows and the last $\rho - 1$ columns.

The matrix now under view is of order $s - \rho + 1$, and, by the theorem already established, the element considered is the product of $(-)^{\frac{1}{2}(s-\rho+1)(s-\rho)+\kappa+1}$ and the determinant of the matrix which is its complementary with regard to the matrix of order $s - \rho + 1$.

If we now form the complementary matrix of $\alpha_{\rho\kappa}$ with regard to the matrix of order s , we find that its determinant has the same numerical value as before, but that its sign is by the added rows and columns multiplied by

$$(-)^{s+s-1+\dots+s-\rho+2} = (-)^{\frac{1}{2}(\rho-1)(2s-\rho+2)};$$

whence the element $\alpha_{\rho\kappa}$ is equal to the determinant of the matrix which is its complement with regard to the matrix of order s multiplied by

$$(-)^{\frac{1}{2}(s-\rho+1)(s-\rho)+\kappa+1+\frac{1}{2}(\rho-1)(2s-\rho+2)},$$

that is, by $(-)^{\frac{1}{2}s(s-1)+\rho+\kappa}$.

Similarly, the α 's are expressed in terms of the α 's, and we may enunciate the general theorem:

“Given a complete table of symmetric functions arranged on the Durfee system, and isolating any square matrix of order s whose secondary diagonal, being coincident with that of the whole square, consists solely of terminal units, the value of any element belonging to the first or second portions of the table and situated in the ρ^{th} row and κ^{th} column is equal to the product of

$$(-)^{\frac{1}{2}s(s-1)+\rho+\kappa}$$

and the determinant of its complementary matrix, when in such matrix all other elements belonging to the first or second portions respectively are replaced by zero.”

The theorem may be otherwise in part exhibited by writing down the identity

$$\begin{vmatrix}
\alpha_{11} & \alpha_{12} & \dots & \alpha_{1, s-3} & \alpha_{1, s-2} & \alpha_{1, s-1} & 1 \\
0 & 0 & \dots & 0 & 0 & 1 & \alpha_{2s} \\
0 & 0 & \dots & 0 & 1 & \alpha_{3, s-1} & \alpha_{3s} \\
0 & 0 & \dots & 1 & \alpha_{4, s-2} & \alpha_{4, s-1} & \alpha_{4s} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 1 & \dots & \alpha_{s-1, s-3} & \alpha_{s-1, s-2} & \alpha_{s-1, s-1} & \alpha_{s-1, s} \\
1 & \alpha_{s2} & \dots & \alpha_{s, s-3} & \alpha_{s, s-2} & \alpha_{s, s-1} & \alpha_{ss}
\end{vmatrix}$$

$$= (-)^{\frac{1}{2}s(s-1)} \{ \alpha_{11}^2 + \alpha_{12}^2 + \dots + \alpha_{1, s-3}^2 + \alpha_{1, s-2}^2 + \alpha_{1, s-1}^2 + 1 \},$$

which follows at once from the foregoing results.

§ 4. We might now investigate a similar theorem for those cases in which the matrix intersects or includes the matrix whose principal diagonal consists entirely of units, and the requisite modification of the determinant rule is easily reached. But, instead of doing this, I will indicate a slight modification of Durfee's arrangement whereby the law above established becomes universally applicable.

It is well known, and moreover very easily proved, that a symmetric function whose partition is self-conjugate contains one, and only one, term whose partition is self-conjugate; the partition of this one term is identical with that of the symmetric function, and, as we know, its coefficient is a terminal unit; it hence follows that the matrix whose principal diagonal consists entirely of terminal units has every other element necessarily zero; as a consequence, if we arrange the self-conjugate partitions of symmetric functions in any order and then place the self-conjugate partitions of the terms in the reverse order, we necessarily confine each portion of the tabulation to half a square and bring all the terminal units into the secondary diagonal.

I therefore propose that Mr. Durfee's arrangement be modified in this manner, so that the self-conjugate partitions are arranged in only a quasi-symmetrical manner. The theorem of § 3 will then be applicable to each of the secondary diagonal matrices of the whole matrix.

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